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The Second-Order Tune Shift with Amplitude for Octupole-induced Resonances in Storage Ring

The purpose of this note is to analyze the octupole-induced resonances, to lowest order, in a synchrotron and storage ring. When the Hamiltonian with octupole term is transformed to action-angle variables, it is found that the amplitude-dependent tune shift terms are composed of two types: terms of second-order in betatron oscillation amplitude of a particle and terms of fourth-order in oscillation amplitude. Obtaining fourth-order terms requires complicated analysis even with the first-order perturbation theory employed. Treatment of this analysis will be the subject of a subsequent note. Second-order terms are straightforward and simple to calculate, and therefore we treat them here first.

The Hamiltonian for a general octupole field in a storage ring is given by:

$$H_1 = \frac{eA_s}{c} = \frac{1}{4!B\rho}\Re e[(\frac{\partial^3 B_y}{\partial x^3} + i\frac{\partial^3 B_x}{\partial y^3})(x+iy)^4] \quad , \tag{1}$$

where A_s is the vector potential and $B\rho$ is the magnetic rigidity.

Normal octupole component:

$$H_1 = \frac{1}{24B\rho}\frac{\partial^3 B_y}{\partial x^3}(x^4 - 6x^2y^2 + y^4) \quad . \tag{2}$$

Skew octupole component:

$$H_1 = \frac{1}{6B\rho}\frac{\partial^3 B_x}{\partial y^3}(x^3y - xy^3) \quad . \tag{3}$$

Total scaled Hamiltonian, including both quadrupole and normal octupole terms, is then given by :

$$H = \frac{p_x^2}{2} + \frac{K_x x^2}{2} + \frac{p_y^2}{2} + \frac{K_y y^2}{2} + \frac{B'''}{24B\rho}(x^4 - 6x^2y^2 + y^4), \tag{4}$$

where $B''' = \partial^3 B_y/\partial x^3$.

The equations of motion corresponding to the octupole term are:

$$\begin{aligned} \Delta x' &= -\frac{B'''l}{6B\rho}(x^3 - 3xy^2) \\ \Delta y' &= \frac{B'''l}{6B\rho}(3x^2y - y^3). \end{aligned} \tag{5}$$

We now perform the canonical transformation to action-angle variables via the generating function:

$$F(x, y, \phi_x, \phi_y; s) = -\frac{x^2}{2\beta_x}(\tan\phi_x + \alpha_x) - \frac{y^2}{2\beta_y}(\tan\phi_y + \alpha_y), \tag{6}$$

where α and β are the usual Twiss parameters:

$$\alpha_z = -\frac{1}{2}\frac{d\beta_z}{ds}, \quad \frac{d\alpha_z}{ds} = \beta_z K_z - \gamma_z \qquad ; z = x, y. \tag{7}$$

The old variables can then be expressed in terms of action and angle variables,

$$\begin{aligned} z &= \sqrt{2\beta_z J_z}\cos\phi_z, \\ p_z &= -\frac{z}{\beta_z}(\tan\phi_z + \alpha_z) = -\frac{\sqrt{2\beta_z J_z}}{\beta_z}\cos\phi_z(\tan\phi_z + \alpha_z). \end{aligned} \tag{8}$$

It is easy to see that the actions J_x and J_y are constants of the motion for the unperturbed Hamiltonian. They are given by:

$$J_z = \frac{(\beta_z p_z + \alpha_z z)^2 + z^2}{2\beta_z} = \frac{\epsilon_z}{2}, \tag{9}$$

where ϵ_z is the emittance of a beam in the z-plane. The new Hamiltonian is then determined from

$$h = H + \frac{\partial F(x, y, \phi_x, \phi_y)}{\partial s} \quad . \tag{10}$$

As a result, the linear term of the new Hamiltonian is

$$h_0 = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y} \tag{11}$$

and the octupole term is

$$\begin{aligned} h_1 =& \frac{B'''}{24B\rho}(x^4 - 6x^2y^2 + y^4) \\ =& \frac{B'''}{24B\rho}[(2\beta_x J_x)^2 \cos^4\phi_x - 6(2\beta_x J_x)(2\beta_y J_y)\cos^2\phi_x \cos^2\phi_y \\ &+ (2\beta_y J_y)^2 \cos^4\phi_y] \equiv V(J_x, J_y, \phi_x, \phi_y; s). \end{aligned} \tag{12}$$

By using

$$\begin{aligned} \cos^4\phi_z =& \frac{\cos 4\phi_z}{8} + \frac{\cos 2\phi_z}{2} + \frac{3}{8} \\ \cos^2\phi_z =& \frac{\cos 2\phi_z + 1}{2} \end{aligned} \tag{13}$$

and

$$\cos 2\phi_x \cos 2\phi_y = \frac{1}{2}[\cos 2(\phi_x + \phi_y) + \cos 2(\phi_x - \phi_y)] \quad , \tag{14}$$

V can be rewritten as:

$$\begin{aligned} V(J, \phi; s) =& \frac{B'''}{48B\rho}[\beta_x^2 J_x^2(\cos 4\phi_x + 4\cos 2\phi_x + 3) \\ &- 6\beta_x\beta_y J_x J_y\{\cos 2(\phi_x + \phi_y) + \cos 2(\phi_x - \phi_y) + 2\cos 2\phi_x \\ &+ 2\cos\phi_y + 2\} + \beta_y^2 J_y^2(\cos 4\phi_y + 4\cos 2\phi_y + 3)]. \end{aligned} \tag{15}$$

In the above equation, the terms that are independent of ϕ introduce the lowest-order tune shift with amplitude (which is the second-order in oscillation amplitude). The ϕ -dependent terms are then the object of the canonical perturbation theory, which leads to the fourth-order tune shift with amplitude. This will be described in a subsequent note. Here we consider the ϕ-independent terms only.

$$V_0 = \frac{B'''}{16B\rho}\beta_x^2 J_x^2 - \frac{B'''}{4B\rho}\beta_x\beta_y J_x J_y + \frac{B'''}{16B\rho}\beta_y^2 J_y^2. \tag{16}$$

From this we can directly extract the tune shifts, which are given by

$$\begin{aligned} 2\pi\Delta\nu_x =& \frac{\partial V_0}{\partial J_x} = +\frac{B'''}{8B\rho}\beta_x^2 J_x - \frac{B'''}{4B\rho}\beta_x\beta_y J_y \\ 2\pi\Delta\nu_y =& \frac{\partial V_0}{\partial J_y} = -\frac{B'''}{4B\rho}\beta_x\beta_y J_x + \frac{B'''}{8B\rho}\beta_y^2 J_y. \end{aligned} \tag{17}$$

In order to relate the above expressions to those given by Collins [1], who reached the same formula by a different approach, we define:

$$\begin{aligned} \underline{m} &\equiv \frac{B'''}{6B\rho}\beta_x^2 = \frac{B'''l}{6B\rho}\delta(s-s_k)\beta_x^2 \\ m &\equiv \frac{B'''}{6B\rho}\beta_x\beta_y = \frac{B'''l}{6B\rho}\delta(s-s_k)\beta_x\beta_y \\ \bar{m} &\equiv \frac{B'''}{6B\rho}\beta_y^2 = \frac{B'''l}{6B\rho}\delta(s-s_k)\beta_y^2 \end{aligned} \tag{18}$$

and

$$a^2 = 2J_x, \quad b^2 = 2J_y \quad . \tag{19}$$

Finally, summing over all the octupoles around the ring, we have:

$$\begin{aligned} 2\pi\Delta\nu_x &= a^2(3/8)\Sigma\underline{m} - b^2(3/4)\Sigma m \\ 2\pi\Delta\nu_y &= -a^2(3/4)\Sigma m + b^2(3/8)\Sigma\bar{m} \quad . \end{aligned} \tag{20}$$

Reference

[1] T.L. Collins, FERMILAB-84/114, October 23, 1984 .